

Lecture 6

Today: * c -convex functions

* Generalized Rockafellar's thm.

As usual, we fix metric spaces X, Y .
Assumed compact, and $c: X \times Y \rightarrow \mathbb{R}$, continuous.

Recall from last lecture:

Definition : A set $\Gamma \subset X \times Y$ is said to be c -cyclically monotone if given any finite family $\{(x_k, y_k)\}_{k=1}^n \subset \Gamma$ and a permutation $\sigma \in S_n$, we have

$$\sum_{k=1}^n c(x_k, y_{\sigma_k}) \geq \sum_{k=1}^n c(x_k, y_k)$$

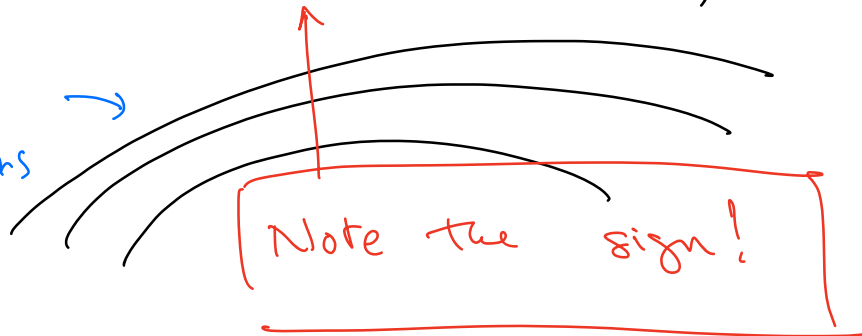
Elements of c -convex analysis

Definition:

A c -function with focus $y \in Y$ is a scalar function on X having the form

$$m(x) = -c(x, y) + \alpha, \quad \alpha \in \mathbb{R}$$

Some
graphs of
 c -functions

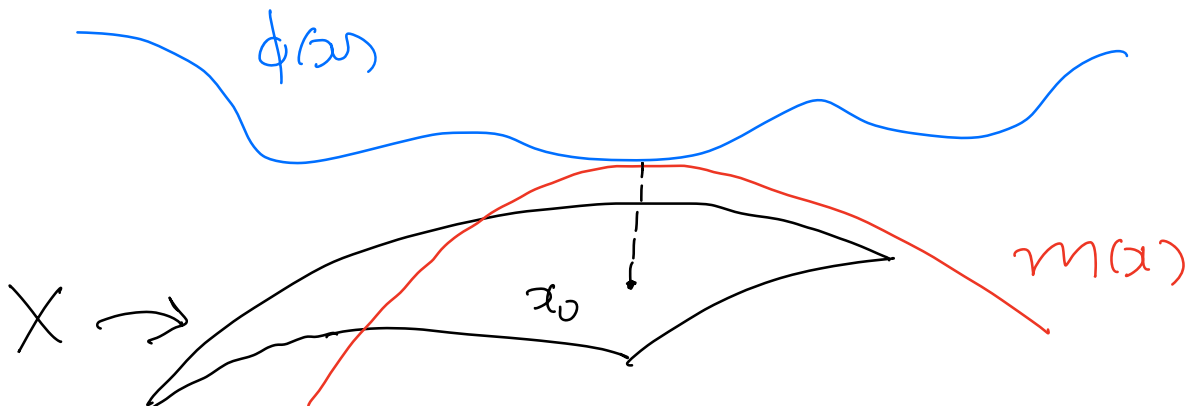


Definition:

A function $\phi: X \rightarrow \mathbb{R}$ is called c -convex if $\forall x_0 \in X$ there exists a c -function

$$m \text{ s.t. } \phi(x) \geq m(x) \quad \forall x \in X$$

$$\phi(x_0) = m(x_0)$$



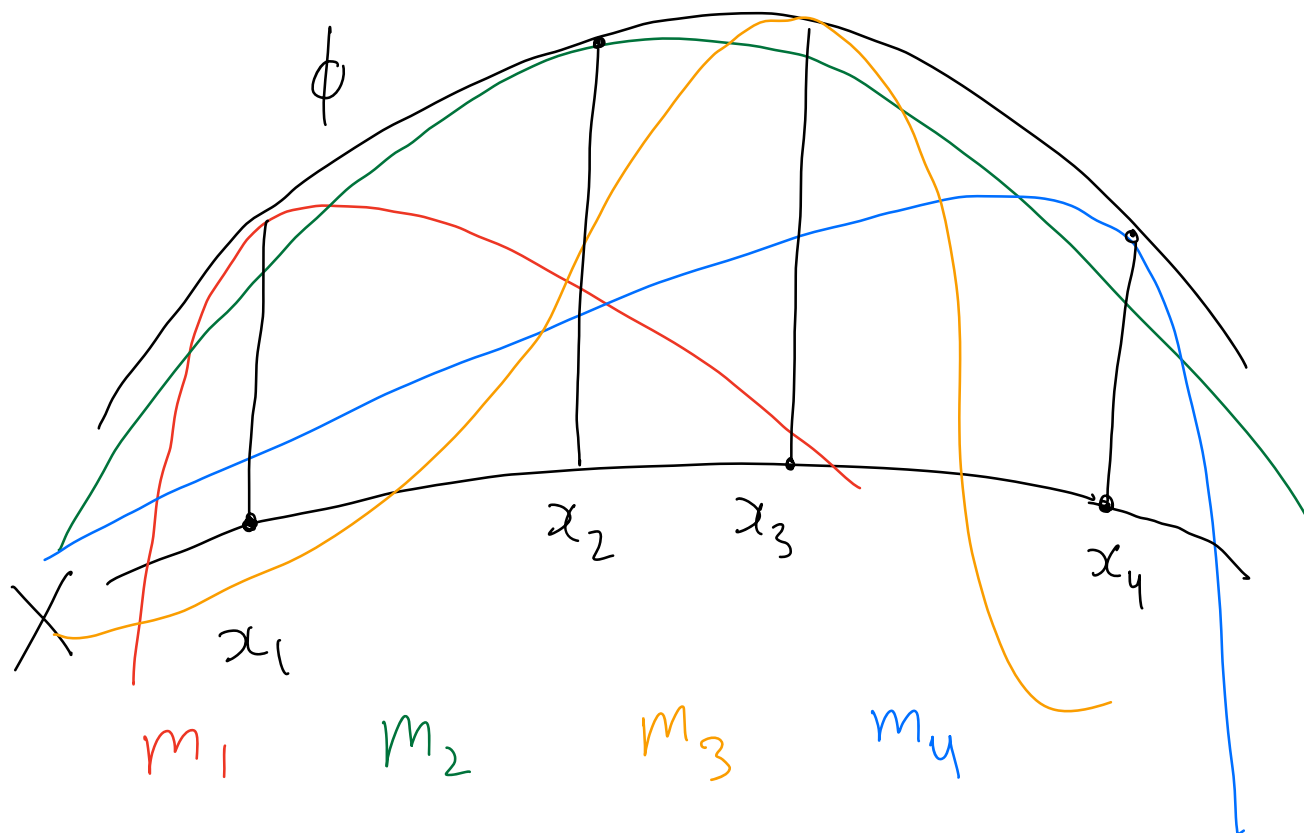
The function is said to be strictly c-convex if $\forall x_0$ and any m as above we have $\phi(x) > m(x)$ when $x \neq x_0$. We say m is a supporting c-function to ϕ at x_0 .

Definition:

If $\phi: X \rightarrow \mathbb{R}$ is c-convex, and $x_0 \in X$, we say $y \in Y$ is in the c-subdifferential of ϕ at x_0 , denoted by $\partial_c \phi(x_0)$, if $\exists$ a c-function m with $\text{ban } y$ which is a supporting c-function to ϕ at x_0 .

Let us suppose $\{(\alpha_k, \gamma_k)\}_{k=0}^n$ is such that $\gamma_k \in \partial_c \phi(\alpha_k) \quad \forall k$, where ϕ is a c -convex function. Then, for each k we have

$$\phi(x) \geq \underbrace{\phi(\alpha_k) - C(x, \gamma_k) + C(\alpha_k, \gamma_k)}_{m_k(x)}$$



We have many inequalities, which we can concatenate to obtain new ones, For instance, we have $\forall x$,

$$\phi(x) \geq \phi(x_1) - C(x, y_1) + C(x_1, y_1)$$

on the other hand, we also have

$$\phi(x_1) \geq \phi(x_2) - C(x_1, y_2) + C(x_2, y_2)$$

and so,

$$\begin{aligned} \phi(x) &\geq \phi(x_2) - C(x, y_1) + C(x_1, y_1) \\ &\quad - C(x_1, y_2) + C(x_2, y_2) \end{aligned}$$

We can do the same with $\phi(x_3), \dots$, and so on until $\phi(x_n)$. This all results in the inequality

$$\begin{aligned} \phi(x) &\geq \phi(x_n) - C(x, y_1) + C(x_1, y_1) \\ &\quad \dots \\ &\quad \dots \\ &\quad - C(x_{n-1}, y_n) + C(x_n, y_n) \end{aligned}$$

which is valid for all $x \in X$.

In particular, the choice $x = x_n$ leads to the inequality

$$0 \geq -c(x_n, y_1) + c(x_1, y_1) - c(x_1, y_2) + c(x_2, y_2) \\ + \dots - c(x_{n-1}, y_n) + c(x_n, y_n)$$

Rearranging, we have (writing $x_0 := x_n$)

$$\sum_{k=1}^n c(x_{k-1}, y_k) \geq \sum_{k=1}^n c(x_k, y_k)$$

This observation proves the following lemma

Lemma: If $\phi: X \rightarrow \mathbb{R}$ is c -convex, the

set $\Gamma_\phi := \{ (x, y) \in X \times Y \mid y \in \partial_c \phi(x) \}$
is c -cyclically monotone.

Of course, we have what is (essentially) the converse statement, which we will call "the generalized Rockafellar theorem"

Theorem: If $\Gamma \subset X \times Y$ is c -cyclically monotone, there exists a c -convex function $\phi : X \rightarrow \mathbb{R}$ s.t.

$$(x, y) \in \Gamma \Rightarrow y \in \partial_c \phi(x).$$

Proof.

Fix $(x_0, y_0) \in \Gamma$.

Consider $\{(x_k, y_k)\}_{k=1}^n$, then, by

assumption

$$\sum_{k=1}^n c(x_{k+1}, y_k) \geq \sum_{k=1}^n c(x_k, y_k) \quad (x_{n+1} = x)$$

$$\text{or,} \quad \sum_{k=1}^n \left\{ -C(x_{k+1}, y_k) + C(x_k, y_k) \right\} \leq 0$$

$$\phi(x) = \sup \sum_{k=1}^n \left\{ -C(x_{k+1}, y_k) + C(x_k, y_k) \right\} \\ \left((x_k, y_k) \dots x_{n+1} = x \right)$$

$$\text{Note :} \quad \phi(x_0) = 0.$$

Observe

$$\begin{aligned} \phi(x) &\leq \sup \left\{ -C(x, y_1) + C(x_0, y_1) \right. \\ &\quad \left. + \dots \right\} \\ &\leq \rho(d(x, x_0)) \quad (\text{or, this is bounded}) \end{aligned}$$

$$\phi(x) - \phi(x') \leq \rho(d(x, x'))$$

... In particular $y_0 \in \text{supp}(\partial_c \phi(x_0))$